Math 656 • FINAL EXAM • May 11, 2010

This is a closed-book exam; neither notes nor electronic devices are allowed. **Please explain all work.**

1) (20pts) Categorize all zeros and singularities of the following functions, find **two** lowest-order non-zero terms in the Laurent or Taylor series of $f(z)$ around the given point z_o, and state the region on which the corresponding expansion is valid. Check your series by calculating the residue at z_o

(a) $f(z)=\dfrac{\sinh z}{1-\cos z}$ at $z_0=0$ (b) $f(z)=\dfrac{\exp(1/z)}{\log_{-\pi} z}$ at $z_0=1$ (branch $\log_{-\pi} z$ satisfies $-\pi \le \arg z < \pi$)

2) (20pts) Describe all singularities of each integrand inside the integration contour, and calculate each integral. Integration contour is a circle of radius 1/2 for both integrals (hint: you may need an inversion mapping in one of these two problems):

(a) $\displaystyle\oint_{|z|=1/2} \frac{z}{\cos(1/z)}dz$ b) $\displaystyle\oint_{|z|=1/2} \frac{\cos(1/z)}{z}dz$

3) (20pts) Calculate any **two** of the following three integrals. Carefully explain each step.

(a) $\displaystyle\int_0^\infty \frac{\cos ax - \cos bx}{x^2}dx = \frac{\pi(b-a)}{2}$ ($a>0, b>0$ are real constants) Use semicircular indented contour

(b) $\displaystyle\int_0^\infty \frac{dx}{x^m+1} = \frac{\pi}{m\sin(\pi/m)}$ ($m>0$ is an integer) Integrate around a circular sector

(c) $\displaystyle\int_0^\infty \frac{\ln x\, dx}{x^2+a^2} = \frac{\pi \ln a}{2a}$ ($a>0$) Intergate $\log_p z/(z^2+a^2)$ around a semi-circular indented contour

4) (20pts) Some of the statements in (a)-(d) below are false. For each false statement, give a counter-example proving that it isn't true. For each true statement, state the theorem from which it follows:

(a) If the integral of $f(z)$ is zero over any closed contour in domain D, then the second derivative of $f(z)$ exists in D, even if D is *not* simply-connected.

(b) If $f(z)$ has a derivative in arbitrary domain D, it must also have an anti-derivative everywhere in D.

(c) Contour integrals of $f(z)$ over two different open contours C_{AB} and C'_{AB} connecting the same end-points A and B are equal if $f(z)$ is analytic along each of these two contours.

(d) Integral of an entire function $f(z)$ over a circle equals twice the integral of $f(z)$ over the corresponding semi-circle.

Choose *one* problem among problems 5-7

5) (20pts) Find coefficients C_{-2} and C_{-4} in the Laurent series for $f(z) = \sec z$ converging within $\pi/2<|z|<3\pi/2$.

6) (20pts) Show that the transformation $w=\dfrac{1}{2}\left(\dfrac{z}{e^a}+\dfrac{e^a}{z}\right)$, where α is a real constant, maps the interior of the unit circle into the exterior of an ellipse.

7) (20pts) Find and sketch the domain of uniform convergence of the series $F(z)=\sum_{n=1}^{\infty}\pi^{-n}\sin nz$ (use exponential representation of $\sin nz$).